

1. Find the derivative

$$(a) f(x) = e^{2 \ln(3x+1)} = e^{\ln(3x+1)^2} = (3x+1)^2$$

$$f'(x) = 2(3x+1)'(3) \text{ or } 18x+6$$

$$(b) f(x) = 5x^{-2} - [\ln \cos x - \ln(\sin x + x)]$$

$$= \frac{-10}{x^3} - \left[\frac{-\sin x}{\cos x} - \frac{\cos x + 1}{\sin x + x} \right]$$

$$(c) f(x) = \frac{e^x + 9}{e^{x^2} - x^4}$$

$$f'(x) = \frac{(e^x - x^4)(e^x) - (2xe^{x^2} - 4x^3)(e^x + 9)}{(e^{x^2} - x^4)^2}$$

$$(d) f(x) = \ln(2x^2 + 1)$$

$$f'(x) = \frac{4x}{2x^2 + 1}$$

$$(e) y = x^{\sqrt{2}}$$

$$y' = \sqrt{2} x^{\sqrt{2}-1}$$

$$(f) y = x^x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \cdot \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x)$$

$$(g) f(x) = \frac{\sec x}{x}$$

$$f'(x) = \frac{x(\sec x \tan x) - 1 \cdot \sec x}{x^2}$$

(h) $\ln y + \overset{\text{P.R.}}{xy^2} - 4x^3 + 10 = 3x$

$$\frac{1}{y} \frac{dy}{dx} + [x(2y) \frac{dy}{dx} + 1 \cdot y^2] - 12x^2 + 0 = 3$$

$$\frac{dy}{dx} \left[\frac{1}{y} + 2xy \right] = 3 - y^2 + 12x^2$$

$$\frac{dy}{dx} = \frac{12x^2 - y^2 + 3}{\frac{1}{y} + 2xy} \quad \text{or} \quad \frac{12x^2y - y^3 + 3y}{1 + 2xy^2}$$

(i) $f(x) = (x^2 + 6) \ln(3x)$

$$f'(x) = (x^2 + 6) \cdot \frac{1}{3x} \cdot 3 + 2x \ln 3x$$

$$\text{or} \quad \frac{3(x^2 + 6)}{3x} + 2x \ln 3x$$

$$\text{or} \quad x + \frac{6}{x} + 2x \ln 3x$$

(j) $f(x) = \cot x$

$$\begin{aligned} \text{or } \frac{d}{dx} \left(\frac{\cos x}{\sin x} \right) &= \frac{\sin x (\sin x) - (\cos x)(\cos x)}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x \end{aligned}$$

(k) $f(x) = x^{\tan x}$

$$y = x^{\tan x}$$

$$\ln y = \tan x \ln x$$

$$\frac{1}{y} y' = \frac{\tan x}{x} + \sec^2 x \ln x$$

$$y' = x^{\tan x} \left(\frac{\tan x}{x} + \sec^2 x \ln x \right)$$

$$\text{or } x^{\tan x - 1} \tan x + x^{\tan x} \sec^2 x \ln x$$

$$\text{or } x^{\tan x - 1} (\tan x + x \sec^2 \ln x) \text{ etc}$$

$$(l) y = \cos x (\tan x - \sec x) = \sin x - 1$$

$$y' = \cos x$$

$$(m) f(x) = 3^{4x}$$

$$f'(x) = 3^{4x} \cdot \ln 3 \cdot 4$$

$$\text{or } 4 \ln 3 \cdot 3^{4x}$$

$$\text{or } \ln 81 \cdot 3^{4x}$$

or if your section 5.5 is rusty:

$$y = 3^{4x}$$

$$\ln y = 4x \ln 3$$

$$\frac{1}{y} y' = 4 \ln 3$$

$$y' = 3^{4x} \cdot 4 \ln 3$$

$$(n) f(t) = \frac{3^{2t}}{t}$$

$$f'(t) = \frac{t(3^{2t} \ln 3 \cdot 2) - 1 \cdot (3^{2t})}{t^2}$$

$$\text{or } \frac{3^{2t} \ln 9}{t} - \frac{3^{2t}}{t^2}$$

$$\text{or } 9^t \left(\frac{\ln 9}{t} - \frac{1}{t^2} \right) \text{ etc...}$$

$$(o) y = \log_5 \frac{x^2 - 1}{x} = \log_5 x^2 - 1 - \log_5 x$$

$$y' = \frac{1}{x^2 - 1} \cdot \frac{1}{\ln 5} \cdot 2x - \frac{1}{x} \cdot \frac{1}{\ln 5}$$

$$\text{or } \frac{1}{\ln 5} \left(\frac{2x}{x^2 - 1} - \frac{1}{x} \right)$$

$$\text{or } \frac{1}{\ln 5} \left(\frac{2x^2 - (x^2 - 1)}{(x^2 - 1)(x)} \right) \text{ or } \frac{1}{\ln 5} \left(\frac{x^2 + 1}{x^2 - x} \right) \text{ etc...}$$

$$(p) g(t) = \log_2(t^2 + 7)^3 = 3 \log_2(t^2 + 7)$$

$$g'(t) = 3 \cdot \frac{1}{t^2 + 7} \cdot (2t) \cdot \frac{1}{\ln 2}$$

$$\text{or } \frac{6t}{\ln 2 (t^2 + 7)}$$

2. Evaluate the integral.

$$(a) \int e^{\sec 2x} \sec 2x \tan 2x \, dx = \frac{1}{2} \int e^u \, du$$

$$\begin{aligned} u &= \sec 2x \\ du &= 2 \sec 2x \tan 2x \, dx \\ \frac{1}{2} du &= \sec 2x \tan 2x \, dx \end{aligned} \quad = \frac{e^{\sec 2x}}{2} + C$$

$$(b) \int \sec y (\tan y - \sec y) \, dy = \int \frac{\sin y}{\cos^2 y} \, dy - \int \sec^2 y \, dy$$

$$\begin{aligned} u &= \cos y \\ du &= -\sin y \, dy \\ -du &= \sin y \, dy \end{aligned}$$

$$= \int u^{-2} \, du - \tan y + C$$

$$= \frac{(\cos y)^{-1}}{-1} - \tan y + C$$

$$\sec y - \tan y + C$$

$$(c) \int e^{3x} \, dx$$

$$\begin{aligned} u &= 3x \\ du &= 3 \, dx \\ \frac{1}{3} du &= dx \end{aligned}$$

$$\frac{1}{3} \int e^u \, du = \frac{1}{3} e^{3x} + C$$

$$(d) \int \tan^2 x + 1 \, dx = \int \sec^2 x \, dx$$

$$\text{Recall } \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan x + C$$

(e) $\int \frac{(\ln x)^2}{x} dx$

$$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array}} \quad = \int u^2 du = \frac{1}{3} u^3 + C = \frac{(\ln x)^3}{3} + C$$

(f) $\int \frac{x}{\sqrt{2x-1}} dx = \frac{1}{2} \int u^{-1/2} \frac{(u+1)}{2} du$

$$\boxed{\begin{array}{l} u = 2x-1 \\ du = 2 dx \\ \frac{1}{2} du = dx \\ x = \frac{u+1}{2} \end{array}} \quad = \frac{1}{4} \int u^{1/2} + u^{-1/2} du$$

$$= \frac{1}{4} \left[\frac{2}{3} u^{3/2} + 2 u^{1/2} \right] + C$$

$$= \frac{1}{6} (2x-1)^{3/2} + \frac{1}{2} \sqrt{2x-1} + C$$

$$\text{or } \frac{\sqrt{2x-1}}{6} (2x-1+3) + C$$

$$\text{or } \frac{1}{3} \sqrt{2x-1} (x+1) + C$$

(g) $\int \frac{1}{3x+2} dx$

$$\boxed{\begin{array}{l} u = 3x+2 \\ du = 3 dx \\ \frac{1}{3} du = dx \end{array}} = \frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln |3x+2| + C$$

(h) $\int \cot x dx = \ln |\sin x| + C \quad (\text{see p 329})$

or $\int \frac{\cos x}{\sin x} dx$

$$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x dx \end{array}}$$

$$\int \frac{1}{u} du$$

$$= \ln |u| + C$$

$$= \ln |\sin x| + C$$

$$(i) \int \frac{12}{1+9x^2} dx = 4 \int \frac{1}{1+u^2} du = \frac{4}{1} \arctan \frac{u}{1} + C$$

$$\begin{array}{l} u = 3x \\ du = 3 dx \\ a = 1 \end{array}$$

$$= 4 \arctan 3x + C$$

$$(j) \int \frac{1}{\sqrt{-x^2-4x}} dx$$

Hint: Complete the square

$$\begin{array}{l} u = x+2 \\ du = dx \end{array}$$

$$-(x^2 + 4x + 4) + 4 = 4 - (x+2)^2$$

$$\int \frac{1}{\sqrt{4-u^2}} du = \arcsin \frac{u}{2} + C = \arcsin \frac{x+2}{2} + C$$

$a=2$

$$(k) \int \frac{e^{2y}}{1-e^{2y}} dy$$

$$\begin{array}{l} u = 1 - e^{2y} \\ du = -2e^{2y} dy \\ -\frac{1}{2} du = e^{2y} dy \end{array}$$

$$-\frac{1}{2} \int \frac{1}{u} du$$

$$-\frac{1}{2} \ln |u| + C$$

$$-\frac{1}{2} \ln |1 - e^{2y}| + C$$

$$(l) \int \frac{e^{3x} - 2e^x + 5}{e^{2x}} dx$$

$$= \int e^{3x-2x} - 2e^{x-2x} + 5e^{-2x} dx$$

$$= \int e^x - 2e^{-x} + 5e^{-2x} dx$$

$$= e^x + 2e^{-x} + \frac{-5}{2} e^{-2x} + C$$

$$(m) \int 2^x dx$$

$$2^x \cdot \frac{1}{\ln 2} + C = \frac{2^x}{\ln 2} + C$$

(n) $\int_1^3 4^{x+1} + 2^x dx$

$$\left. \frac{4^{x+1}}{\ln 4} + \frac{2^x}{\ln 2} \right|_1^3 = \left. \frac{4 \cdot 4^x}{2 \cdot \ln 2} + \frac{2^x}{\ln 2} \right|_1^3 = \left. \frac{2 \cdot 4^x + 2^x}{\ln 2} \right|_1^3 = \frac{1}{\ln 2} \left[(128 + 8) - (8 + 2) \right] = \frac{126}{\ln 2}$$

(o) $\int_1^3 \frac{e^{3/x}}{x^2} dx = \frac{1}{3} \int_3^1 e^u du = \frac{1}{3} \int_1^3 e^u du = \frac{1}{3} e^u \Big|_1^3$

$$u = \frac{3}{x} \quad \quad \quad = \frac{e^3 - e^1}{3}$$

$$du = -\frac{3}{x^2} dx$$

$$-\frac{1}{3} du = \frac{1}{x^2} dx$$

$$u(1) = 3$$

$$u(3) = 1$$

(p) $\int_0^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}} dx = \arcsin \frac{x}{2} \Big|_0^{\sqrt{2}} = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

$$a = 2$$

(q) $\int_{-2}^3 \frac{1}{x^2 + 4x + 8} dx = \int_{-2}^3 \frac{1}{(x^2 + 4x + 4) + 4} dx = \int_{-2}^3 \frac{1}{(x+2)^2 + 2^2} dx$

Hint: Complete the square

$$\begin{array}{l} a=2 \\ u=x+2 \\ u(3)=5 \\ u(-2)=0 \end{array} \quad \frac{1}{2} \arctan \frac{u}{2} \Big|_0^5$$

$$\frac{1}{2} \arctan \frac{x+2}{2} \Big|_{-2}^3 = \frac{1}{2} \left[\arctan \frac{5}{2} - \arctan 0 \right] = \frac{1}{2} \arctan \frac{5}{2}$$

(r) $\int_0^{\pi/2} \frac{\cos x}{2^{\sin x}} dx = \int_0^1 2^{-u} du = \left. \frac{-1}{\ln 2} 2^{-u} \right|_0^1 = \frac{-1}{\ln 2} \left(\frac{1}{2} - 1 \right) = \frac{1/2}{\ln 2} = \frac{1}{2 \ln 2} = \frac{1}{\ln 4}$

$$\begin{array}{l} u = \sin x \\ du = \cos x dx \\ u(0) = 0 \\ u(\frac{\pi}{2}) = 1 \end{array}$$

$$\int_0^1 \left(\frac{1}{2}\right)^u du = \left. \frac{1}{1-\frac{1}{2}} \left(\frac{1}{2}\right)^u \right|_0^1 = \frac{1}{1/2} \left(\frac{1}{2} - 1 \right) = \frac{-1/2}{1/2} = \frac{-1/2}{1/2} = \frac{1}{2 \ln 2} = \frac{1}{\ln 4} \approx .721$$

3. Evaluate the limits, using L'Hôpital's Rule if necessary. If you do, remember to identify if it is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and state that you are using L'Hôpital's Rule.

(a) $\lim_{x \rightarrow -3} \frac{3 \sin(2x+6)}{3+x}$ $\frac{0}{0}$ form, by L'Hôpital: $\lim_{x \rightarrow -3} \frac{3(2) \cos(2x+6)}{1} = 6$

(b) $\lim_{x \rightarrow 3} \frac{3 \ln(4-x)}{x-3}$ $\frac{0}{0}$ form, by L'Hôpital: $\lim_{x \rightarrow 3} \frac{\frac{3}{4-x}(-1)}{1} = -3$

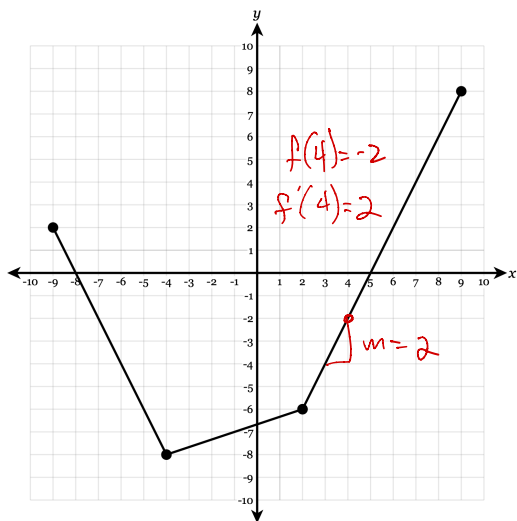
(c) $\lim_{x \rightarrow \infty} \frac{\arctan x}{3} = \frac{\frac{\pi}{2}}{3} = \frac{\pi}{6}$ (Do not use L'Hôpital)

(d) $\lim_{x \rightarrow 2} \frac{x^2-4}{x+2} = \frac{0}{4} = 0$ (Do not use L'Hôpital)

(e) $\lim_{x \rightarrow \infty} \frac{\ln x^2}{(\ln x)^2}$ $\frac{\infty}{\infty}$ form, by L'Hôpital: $\lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} \cdot 2x}{2 \ln x \cdot \frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{2}{x}}{\frac{2}{x} \ln x} = 0$

(f) $\lim_{x \rightarrow \infty} \frac{\ln 6x}{\ln 2x}$ $\frac{\infty}{\infty}$ form, by L'Hôpital: $\lim_{x \rightarrow \infty} \frac{\frac{1}{6x} \cdot 6}{\frac{1}{2x} \cdot 2} = 1$

4. The graph of the function f is shown below. Determine the value of $\lim_{x \rightarrow 2} \frac{f(2x)+2}{5x-10}$



$f(4)+2 = -2+2=0$ $\frac{0}{0}$ form so by L'Hôpital:
 $5(2)-10=0$

$= \lim_{x \rightarrow 2} \frac{f'(2x) \cdot 2}{5}$ ← remember Chain Rule!

$f'(4) = 2$

So $\lim_{x \rightarrow 2} \frac{f'(2x)}{5} = \frac{4}{5}$

(See Practice L'Hôpital at Delta Math for more like this)

5. Find an equation of the tangent line to $y = 5^{x-2}$ at the point $(2, 1)$

$$\begin{aligned} y' &= (\ln 5)(5^{x-2})(1) \\ y'(2) &= (\ln 5)(5^0) = \ln 5 \\ y - 1 &= \ln 5(x - 2) \end{aligned}$$

6. If $f(x) = \int_{\arctan x}^2 7^t dt$, then find $f'(x)$. (Hint: FTC2 and the chain rule)

$$\begin{aligned} f(x) &= - \int_2^{\arctan x} 7^t dt \quad \text{by 2nd FTC} \\ f'(x) &= - 7^{\arctan x} \left(\frac{1}{x^2+1} \right) = - \frac{7^{\arctan x}}{x^2+1} \end{aligned}$$

7. (Calculator Active) The weight (in grams) of a bacterial culture at time t (hours) is modeled by the function

$$W(t) = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

for time $t \geq 0$

- (a) Find the weight after 1 hour.

$$W(1) = 1.07059044 \text{ grams}$$

- (b) Find the rate at which the weight is increasing after 2 hours

$$\begin{aligned} \left. \frac{d}{dt} W(t) \right|_{t=2} &= 0.0453947242 \text{ grams per hour} \\ &\quad \text{(increasing)} \\ &\quad \text{(use MATH 8.n Deriv on TI)} \end{aligned}$$

8. (Calculator Active) At what point (x, y) on the graph of $y = 2^x - 3$ does the tangent line have slope 21?

$$\begin{aligned} y' &= \ln 2 \cdot 2^x = 21 \\ 2^x &= \frac{21}{\ln 2} \\ x &= \log_2 \left(\frac{21}{\ln 2} \right) \\ x &= \frac{\ln \left(\frac{21}{\ln 2} \right)}{\ln 2} \approx 4.921083796 \\ \text{point is } &(4.921, 27.296) \text{ or } (4.921, 27.297) \end{aligned}$$

9. (No Calculator) A particle moves along the x axis so that at any time $t > 0$ its velocity is given by $v(t) = t \ln t - t$. At time $t = 1$, the position of the particle is $x(1) = 6$.

product rule

- (a) Write an expression for the acceleration of the particle.

$$a(t) = v'(t) = (t)\left(\frac{1}{t}\right) + (1)(\ln t) - 1$$

$$= 1 + \ln t - 1$$

$$= \ln t$$


- (b) For what values of t is the particle moving right?

when $v(t) > 0$

$$t \ln t - t > 0$$

$$t(\ln t - 1) > 0$$

given $t > 0$, so when $\ln t - 1 > 0$
 $\ln t > 1$
 $t > e^1$

- (c) What is the minimum velocity of the particle. Justify your conclusion.



so $v'(t)$ changes from neg.

to pos at $t = 1$

so min velocity is $1(\ln 1 - 1) = 0 - 1 = -1$

- (d) If $\int t \ln t - t \, dt = \frac{1}{4}t^2(2 \ln t - 3) + C$, write an expression of the position $x(t)$ of the particle.

$$x(1) = 6 \quad \text{so}$$

$$\frac{1}{4}1^2(2 \ln 1 - 3) + C = 6$$

$$\frac{1}{4}(-3) + C = 6$$

$$C = 6 + \frac{3}{4} = \frac{24+3}{4} = \frac{27}{4}$$

$$x(t) = \frac{1}{4}t^2(2 \ln t - 3) + \frac{27}{4}$$

10. (No Calculator) Let $f(x) = e^x \cos x$.

(a) (1 point) Find the average rate of change of f on the interval $0 \leq x \leq \pi$.

$$\begin{aligned} \frac{f(\pi) - f(0)}{\pi - 0} &= \frac{e^\pi \cos \pi - e^0 \cos 0}{\pi} \\ &= \frac{e^\pi(-1) - 1}{\pi} \left(-\frac{e^\pi}{\pi} - \frac{1}{\pi} \right) \end{aligned}$$

(b) (2 points) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

$$\begin{aligned} f'(x) &= e^x(-\sin x) + e^x \cos x = e^x(\cos x - \sin x) \\ f'\left(\frac{3\pi}{2}\right) &= e^{3\pi/2} \left(\cos \frac{3\pi}{2} - \sin \frac{3\pi}{2} \right) \\ &= e^{3\pi/2} (0 - (-1)) = e^{3\pi/2} \end{aligned}$$

(c) (3 points) Find the absolute minimum value of f on the interval $0 \leq x \leq 2\pi$. Justify your answer.

Candidate test

$$0 = f'(x) = e^x (\cos x - \sin x)$$

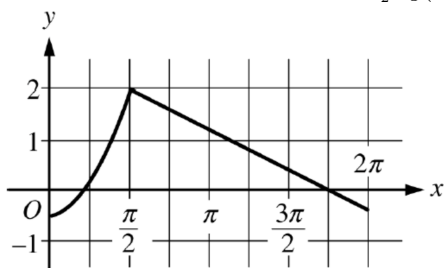
$$\cos x = \sin x$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4} \quad (\text{when})$$

x	0	$\frac{\pi}{4}$	$\frac{5\pi}{4}$	2π
$f(x)$	1	$e^{1/4} \cdot \frac{\sqrt{2}}{2}$	$e^{5\pi/4} \cdot \frac{-\sqrt{2}}{2}$	$e^{2\pi}$

Neg. Abs Min by Candidate Test

(d) (3 points) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g' , the derivative of g , is shown below. Find the value of $\lim_{x \rightarrow \frac{\pi}{2}} \frac{f(x)}{g(x)}$ and therefore continuous.



Graph of g'

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= e^{\pi/2} \cdot \cos \frac{\pi}{2} = 0 \\ g\left(\frac{\pi}{2}\right) &= 0 \quad (\text{given}) \quad \text{so } \frac{0}{0} \text{ form} \\ \text{by L'H: } \lim_{x \rightarrow \frac{\pi}{2}} \frac{f'(x)}{g'(x)} &= \frac{e^{\pi/2}(\cos \frac{\pi}{2} - \sin \frac{\pi}{2})}{2} \\ &= \frac{-e^{\pi/2}}{2} \end{aligned}$$